

THE VANISHING OF THE CONTACT INVARIANT IN THE PRESENCE OF TORSION

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ABSTRACT. We prove that the Ozsváth-Szabó contact invariant of a closed 3-manifold with positive 2π -torsion vanishes.

In 2002, Ozsváth and Szabó [OSz1] defined an invariant of a closed, oriented, contact 3-manifold (M, ξ) as an element of the Heegaard Floer homology group $\widehat{HF}(-M)$. The definition of the contact invariant was made possible by the work of Giroux [Gi3], which related contact structures and open book decompositions. The Ozsváth-Szabó contact invariant has undergone an extensive study, e.g., [LS1, LS2]. Recently, Honda, Kazez and Matić [HKM3] defined an invariant of a contact 3-manifold with convex boundary as an element of Juhász' sutured Floer homology [Ju1, Ju2]. The goal of this paper is to use this relative contact invariant to prove a vanishing theorem in the presence of torsion.

Recall that a contact manifold (M, ξ) has *positive $n\pi$ -torsion* if it admits an embedding $(T^2 \times [0, 1], \eta_{n\pi}) \hookrightarrow (M, \xi)$, where (x, y, t) are coordinates on $T^2 \times [0, 1] \simeq \mathbb{R}^2/\mathbb{Z}^2 \times [0, 1]$ and $\eta_{n\pi} = \ker(\cos(n\pi t)dx - \sin(n\pi t)dy)$. The torsion was an essential ingredient for distinguishing tight contact structures on toroidal 3-manifolds (see for example [Gi1]), and is a source of non-finiteness of the number of isotopy classes of tight contact structures ([CGH, Co, HKM1]).

Theorem 1 (Vanishing Theorem). *If a closed contact 3-manifold (M, ξ) has positive 2π -torsion, then its contact invariant $c(M, \xi)$ in $\widehat{HF}(-M)$ vanishes.*

The coefficient ring of $\widehat{HF}(-M)$ is \mathbb{Z} in Theorem 1. The behavior of the contact invariant with twisted coefficients in presence of torsion is the subject of a forthcoming paper by the first two authors [GH].

Theorem 1 was first conjectured in [Gh2, Conjecture 8.3], and partial results were obtained by [Gh1], [Gh2], and [LS3]. The corresponding vanishing result for the contact class in monopole Floer homology has recently been announced by Mrowka and Rollin (and is motivated by [Ga]). Theorem 1, together with a non-vanishing result of the contact invariant proved by Ozsváth and Szabó [OSz2, Theorem 4.2], implies that a contact manifold with positive

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2π -torsion is not strongly symplectically fillable. This non-fillability result was conjectured by Eliashberg, and first proved by Gay [Ga].

In this paper, a contact structure ξ on a compact, oriented 3-manifold N with convex boundary ∂N and dividing set Γ on ∂N will be denoted (N, Γ, ξ) . We will write the invariant for a closed contact 3-manifold (M, ξ) as $c(M, \xi) \in \widehat{HF}(-M)$ and the invariant for a compact contact 3-manifold (N, Γ, ξ) as $c(N, \Gamma, \xi) \in SFH(-N, -\Gamma)$, where $SFH(-N, -\Gamma)$ is the sutured Floer homology of $(-N, -\Gamma)$, and $\Gamma \subset \partial N$ is now viewed as a balanced suture. Strictly speaking, the contact invariants have a ± 1 ambiguity, but this will not complicate matters in this paper. The key property of the relative contact invariant which we use in this paper is the following theorem from [HKM3]:

Theorem 2 ([HKM3, Theorem 4.5]). *Let (M, ξ) be a closed contact 3-manifold and $N \subset M$ be a compact submanifold (without any closed components) with convex boundary and dividing set Γ . If $c(N, \Gamma, \xi|_N) = 0$, then $c(M, \xi) = 0$.*

The behavior of the contact invariant under contact surgery will also play a fundamental role in the proof of Theorem 1.

Lemma 3. *If (N', Γ', ξ') is obtained by contact $(+1)$ -surgery on a Legendrian curve in (N, Γ, ξ) , then the contact $(+1)$ -surgery gives rise to a natural map:*

$$(1) \quad \Phi: SFH(-N, -\Gamma) \rightarrow SFH(-N', -\Gamma'),$$

which satisfies $\Phi(c(N, \Gamma, \xi)) = c(N', \Gamma', \xi')$.

Proof. If (N', ξ') is obtained from (N, ξ) by contact $(+1)$ -surgery, then (N, ξ) is obtained from (N', ξ') by contact (-1) -surgery (i.e., Legendrian surgery); see [DG1, Proposition 8]. The proof that the contact invariant is natural with respect to Legendrian surgery is the same as in the closed case, provided we use the reformulation of the contact invariant given by Honda, Kazez, and Matić [HKM2]. The proof in the closed case is given in [HKM2, Proposition 3.7]. See also [HKM3, Proposition 4.4]. \square

In this paper we assume that the reader is familiar with the terminology introduced in [H1], such as *basic slice*, *standard neighborhood of a Legendrian curve*, *Legendrian ruling curve*, and *minimally twisting*.

Let Γ be the following suture/dividing set on the boundary of $T^2 \times [0, 1]$: $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$, $\text{slope}(\Gamma_{T_0}) = -1$, and $\text{slope}(\Gamma_{T_1}) = -2$. Here $\#$ denotes the number of connected components, $T_i = T^2 \times \{i\}$, the slope is calculated with respect to a fixed oriented identification $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$, and the orientation of T_i is inherited from that of T^2 . (Hence $\partial(T^2 \times [0, 1]) = T_1 \cup -T_0$.)

Let ζ_0 be a tight contact structure so that $(T^2 \times [0, 1], \Gamma, \zeta_0)$ is a basic slice. There are two possible isotopy classes rel boundary, and ζ_0 can be in either one.

Lemma 4. *Let L be a Legendrian ruling curve with infinite slope on a parallel copy T_ε of T_0 with the same dividing set, inside the basic slice $(T^2 \times [0, 1], \Gamma, \zeta_0)$. Then there is an embedding i of $(T^2 \times [0, 1], \Gamma, \zeta_0)$ into the standard tight (S^3, ξ_{std}) , so that $i(L)$ is an unknot with Thurston-Bennequin invariant -1 .*

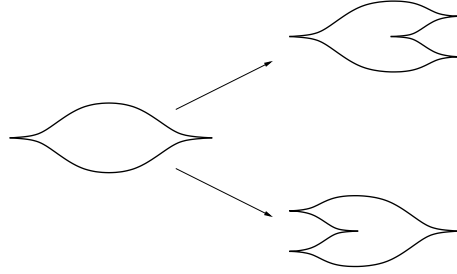


FIGURE 1. Positive and negative stabilizations of the Legendrian unknot in S^3 with Thurston–Bennequin number -1 .

Proof. Choose coordinates (x, y) on $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ and z on $[0, 1]$. Then $(T^2 \times [0, 1], \Gamma, \zeta_0)$ is contact isomorphic to a basic slice with boundary slopes $-\frac{1}{2}$ and -1 via the diffeomorphism $(x, y, z) \mapsto (y, x, 1 - z)$. Under this diffeomorphism L is mapped to a curve with slope 0 .

Let V be a standard neighborhood of a Legendrian unknot K in (S^3, ξ_{std}) with Thurston–Bennequin number -1 . Then $\text{slope}(\Gamma_{\partial V}) = -1$ and $\#\Gamma_{\partial V} = 2$, where the slopes are computed with respect to a basis on $H_1(\partial V)$ such that the meridian has slope 0 and the longitude determined by the Seifert surface has slope ∞ . If we stabilize K and let V' be a sufficiently small standard neighborhood of the stabilized curve, then the collar region $V \setminus V'$ is a basic slice with boundary slopes $-\frac{1}{2}$ and -1 . Recall that K can be stabilized in two different ways, which correspond to two different basic slices — it is easy to relate the relative Euler class of the basic slice with the rotation number of the stabilized knot. See Figure 1 for the two different stabilizations of K , drawn in the front projection.

The basic slice $(T^2 \times [0, 1], \Gamma, \zeta_0)$ with boundary slopes -1 and -2 is contact isomorphic to the basic slice $(V \setminus V', \Gamma_{\partial V'} \cup \Gamma_{\partial V}, \xi_{std}|_{V \setminus V'})$, and the Legendrian knot $L \subset T^2 \times [0, 1]$ corresponds to a pushoff of the meridian of V . Therefore, the image of L is an unknot, and the Thurston–Bennequin invariant is easily calculated from the number of intersections with $\Gamma_{\partial V}$. \square

Lemma 5. *The contact manifold (M, ξ) has positive 2π -torsion if and only if there exists an embedding of $(T^2 \times [0, 1], \Gamma, \zeta_1)$ into (M, ξ) , where $(T^2 \times [0, 1], \Gamma, \zeta_1)$ is not minimally twisting and is homotopic relative to the boundary to a basic slice $(T^2 \times [0, 1], \Gamma, \zeta_0)$.*

Proof. From the classification of tight contact structures on $T^2 \times [0, 1]$ (see Theorem 2.2 as well as the discussion in Section 5.2 in [H1]; an equivalent result is given in [Gi2]) it follows that, if ζ_1 is not minimally twisting and is homotopic to a basic slice, then $(T^2 \times [0, 1], \zeta_1)$ has positive 2π -torsion. Therefore the existence of an embedding of $(T^2 \times [0, 1], \zeta_1)$ into (M, ξ) implies that (M, ξ) has positive 2π -torsion.

Assume (M, ξ) contains a contact submanifold isomorphic to $(T^2 \times [0, 1], \eta_{2\pi})$. Then it also contains a slightly larger submanifold (N, ζ') , where $N = T^2 \times [-\varepsilon_0, 1 + \varepsilon_1]$, and ζ' is defined by the same contact form as $\eta_{2\pi}$. This can be easily seen from the normal form of a contact structure in the neighborhood of a pre-Lagrangian torus. By direct computation, we can choose $\varepsilon_0, \varepsilon_1 \geq 0$ so that the tori $T^2 \times \{-\varepsilon_0\}$ and $T^2 \times \{1 + \varepsilon_1\}$ are pre-Lagrangian tori with rational slopes s_1, s_2 forming an integer basis of $H_1(T^2)$. Then we can perturb the boundary of N to make it convex, so that the boundary tori have $\#\Gamma = 2$ and slopes s_1, s_2 ;

see for example [Gh3, Lemma 3.4]. Let ζ_1 be the resulting contact structure: the contact manifold (N, ζ_1) constructed in this way is clearly non-minimally-twisting. After a change of coordinates in N , we can make its boundary slopes -1 and -2 . The contact structure is homotopic to a basic slice by a standard explicit computation (see [Gh2, Proposition 6.1]). \square

Proof of Theorem 1. By Theorem 2 and Lemma 5, it suffices to prove that $c(N, \Gamma, \zeta_1) = 0$, where $N = T^2 \times [0, 1]$ and Γ, ζ_1 are as defined above. This proof is modeled on the argument in [Gh2].

Take a parallel copy T_ε of T_0 in the interior of N with the same dividing set, and let L be a Legendrian ruling curve on T_ε with slope ∞ . The Legendrian curve L has twisting number -1 with respect to the framing coming from T_ε .

Now apply a contact $(+1)$ -surgery to N along L ; see for example [DG2]. As the surgery coefficient is 0 with respect to the framing induced by the torus T_ε , the resulting 3-manifold is $N' = (S^1 \times D^2) \# (S^1 \times D^2)$. Next write Γ' as $\Gamma'_1 \sqcup \Gamma'_2$, where Γ'_i is the dividing set on the i th connect summand $S^1 \times D^2$. Since each component of Γ'_i intersects the meridian once geometrically, we may take Γ'_i to have slope ∞ , after diffeomorphism. (Here the slope of the boundary of a solid torus is defined by setting the meridian to have slope 0 and choosing some longitude.)

It was proved in [HKM3] that $SFH(-N, -\Gamma) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, where each \mathbb{Z} -summand corresponds to a distinct relative Spin^c -structure. As for $SFH(-N', -\Gamma')$, Juhász [Ju1, Proposition 9.15] proved that the sutured Floer homology of a connected sum of two balanced sutured manifolds is the sutured Floer homology of their tensor product, tensored with an extra \mathbb{Z}^2 factor. Since each $(S^1 \times D^2, \Gamma'_i)$ is product disk decomposable, $SFH(S^1 \times D^2, \Gamma'_i) \cong \mathbb{Z}$, and hence $SFH(-N', -\Gamma') \cong (\mathbb{Z} \otimes \mathbb{Z}) \otimes \mathbb{Z}^2 \cong \mathbb{Z}^2$.

Let \mathfrak{s} be the relative Spin^c -structure induced by ζ_1 . We claim that the map Φ induced by the surgery is injective on the direct summand $SFH(-N, -\Gamma, \mathfrak{s}) \cong \mathbb{Z}$; that is the content of Lemma 6 below. In Lemma 7, we will prove that applying contact $(+1)$ -surgery to (N, Γ, ζ_1) along L yields an overtwisted contact structure ζ'_1 on N' . Therefore, $\Phi(c(N, \Gamma, \zeta_1)) = c(N', \Gamma', \zeta'_1) = 0$, and by the injectivity of Φ on the appropriate \mathbb{Z} -summand it follows that $c(N, \Gamma, \zeta_1) = 0$. \square

Lemma 6. *Let \mathfrak{s} be the relative Spin^c -structure induced by (Γ, ζ_1) and \mathfrak{s}' be that induced by (Γ', ζ'_1) . Then the map*

$$\Phi: SFH(-N, -\Gamma, \mathfrak{s}) \rightarrow SFH(-N', -\Gamma', \mathfrak{s}'),$$

given by Equation 1, is injective.

Proof. Recall that ζ_0 and ζ_1 have the same relative Spin^c -structure \mathfrak{s} . By Lemma 4, (N, Γ, ζ_0) can be embedded in (S^3, ξ_{std}) , which has nonzero contact invariant. Hence, by Theorem 2, the contact invariant $c(N, \Gamma, \zeta_0) \in SFH(-N, -\Gamma, \mathfrak{s})$ is nonzero. Since $SFH(-N, -\Gamma, \mathfrak{s}) \cong \mathbb{Z}$ (since it is nonzero) and $SFH(-N', -\Gamma') \cong \mathbb{Z}^2$, it suffices to prove that $\Phi(c(N, \Gamma, \zeta_0)) \neq 0$.

By Lemma 3, the cobordism map Φ takes the contact class $c(N, \Gamma, \zeta_0)$ to $c(N', \Gamma', \zeta'_0)$, where ζ'_0 is the contact structure obtained from ζ_0 by contact $(+1)$ -surgery along L . Now consider the embedding $i: (N, \Gamma, \zeta_0) \hookrightarrow (S^3, \xi_{std})$ from Lemma 4. Legendrian $(+1)$ -surgery along the unknot $i(L)$ with Thurston-Bennequin invariant -1 inside (S^3, ξ_{std}) yields the

unique tight contact structure on $S^1 \times S^2$, which has nonzero contact invariant: for example, see [LS2, Lemma 3.7]. Hence $c(N', \Gamma', \zeta'_0) \neq 0$, and it follows that $SFH(-N, -\Gamma, \mathfrak{s})$ maps injectively into $SFH(-N', -\Gamma')$. \square

Lemma 7. *Applying contact $(+1)$ -surgery to (N, Γ, ζ_1) along L yields an overtwisted contact structure ζ'_1 on N' .*

Proof. For any $s \in \mathbb{Q} \cup \{\infty\}$, there is a convex torus (in standard form) with slope s in (N, Γ, ζ_1) parallel to the boundary, according to [H1, Proposition 4.16]. In particular, there is a standard torus whose Legendrian divides have the same slope as the Legendrian ruling curve L we are doing surgery on. After the surgery, this Legendrian divide bounds an overtwisted disk in N' . \square

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